Painlevé Classification of Binomial Type Ordinary Differential Equations of the Arbitrary Order

By S. Sobolevsky

The complete Painlevé classification of the binomial ordinary differential equations of the arbitrary order $n \geq 4$ is built. Six classes of equations with Painlevé property are obtained. All of these equations are solved in terms of elementary functions and known Painlevé transcendents.

1. Introduction

Binomial type ordinary differential equations are the equations of the form

$$\left(y^{(n)}\right)^m = R\left(y^{(n-1)}, y^{(n-2)}, \ldots, y', y, x\right), \tag{1}$$

where n and m are natural numbers, $m > 1$ and R is a rational function in y and its derivatives, locally analytic in x. The Painlevé classification of the first-order equations (1) is well known [1]. The solution of the second-order problem has been completed by C. M. Cosgrove and G. Scoufis in [2, 3]. The third-order problem in case of $m > 1$ and R not being an exact mth power has been solved in [4]. In the present paper we solve the higher-order problem in the same assumption: $m > 1$ and R is not an exact mth power. Note that we do not consider a case $m = 1$ which corresponds the rational differential equations. In spite of their simpler analytical structure, such equations are

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much harder to investigate. Some partial results for them have been obtained in a number of works, particularly in [5–12].

2. Analytical structure of *R*

Assume that Equation (1) of order $n \geq 4$ is free of movable critical singularities and that $m > 1$, while for any $m_1 \mid m$, R is not an exact m_1th power of a function rational in $y^{(n-1)}$.

First, note that R should be polynomial in $y^{(n-1)}$. Otherwise let R admit a pole of order p along a curve $y^{(n-1)} = h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)$, where h is an analytic function in all variables in a neighborhood of a certain point $(y_0^{n-2}, y_0^{n-3}, \ldots, y_0^0, x_0)$. That is, let R be of the following form $R(y^{(n-1)}, y^{(n-2)}, \ldots, y, x) = (y^{(n-1)} - h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x))^{-p} R_1(y^{(n-1)}, y^{(n-2)}, \ldots, y, x)$, where R_1 is a function, analytic and nonzero in a neighborhood of the point $(y_0^{n-1}, y_0^{n-2}, \ldots, y_0^0, x_0)$, while $y_0^{n-1} = h(y_0^{n-2}, y_0^{n-3}, \ldots, y_0^0, x_0)$. Establish a transformation $x = x_0 + \alpha^{p+m} X$, $y^{(j)} = y_0^j + \alpha^{p+m} v^j$, $j = \overline{0..n-2}$, $y^{(n-1)} = y_0^{n-1} + \alpha^m v^{n-1}$. Obtain a reduced for $\alpha = 0$ differential equation $(v^{n-1})' = (v^{n-1})^{-p/m} \sqrt[m]{R_1(y_0^{n-1}, y_0^{n-2}, \ldots, y_0^0, x_0)}$. This equation admits movable critical singularities. So the initial Equation (1) in the considered case also admits movable critical singularities.

Let $d = \deg_{y^{(n-1)}} R$ and $R = \sum_{j=0}^{d} R_j(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)(y^{(n-1)})^j$, where functions R_j are rational in y and its derivatives and locally analytic in x. Note that $d \leq 2m$. Otherwise, introduce a transformation $x = x_0 + \alpha^{d-m} x$, $y^{(j)} = y_0^j + \alpha^{d-m} v^j$, $j = \overline{0..n-3}$, $y^{(n-2)} = y_0^{n-2} + \alpha^{d-2m} u$. Obtain a reduced differential equation $u'' = (u')^{d/m} \sqrt[m]{R_d(y_0^{n-2}, y_0^{n-3}, \ldots, y_0^0, x_0)}$ for $\alpha = 0$. This equation admits movable critical singularities, so the initial Equation (1) in the case $d > 2m$ also admits them.

If $d = 2m$, then introducing a transformation $x = x_0 + \alpha^m x$, $y^{(j)} = y_0^j + \alpha^m v^j$, $j = \overline{0..n-3}$, $y^{(n-2)} = u$, obtain a reduced differential equation of the form $u'' = \epsilon (u')^2 \sqrt[m]{R_d(u, y_0^{n-3}, y_0^{n-4}, \ldots, y_0^0, x_0)}$, where ϵ is an arbitrary root of an equation $\epsilon^m = 1$. According to Cosgrove [3], this equation admits movable critical singularities for at least one value of ϵ. So the case $d = 2m$ is also impossible.

We proved that R should be a polynomial in $y^{(n-1)}$ of degree $d < 2m$.

Now, consider the factorization of the rational function R and for that purpose represent it in the following form:

$$R\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big) = T\big(y^{(n-2)}, y^{(n-3)}, \ldots, y, x\big) \times \prod_{j=1}^{s} \big(P_j(y^{(n-1)}, y^{(n-2)}, \ldots, y, x)\big)^{g_j}, \quad (2)$$

where s is a positive integer; g_j are natural numbers; P_j are irreducible pairwise coprime polynomials in $y, y', \ldots, y^{(n-1)}$ with coefficients locally analytic in x; $p_j = \deg_{y^{(n-1)}} P_j > 0$; T is a rational function in $y, y', \ldots, y^{(n-2)}$ locally analytic in x.

We obtain necessary conditions for the numbers g_j and p_j as well as for the polynomials p_j structure. Consider a polynomial P_k for an arbitrary k and assume that g_k/m is not an integer.

Let

$$H\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big) = \sqrt[m]{R\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big)/P_k\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big)^{g_k}}$$

and consider an arbitrary point $\lambda = (y_0^{n-1}, y_0^{n-2}, \ldots, y_0, x_0)$, such that $P_k(\lambda) = 0$ and $H(\lambda) \neq 0$. Introduce to Equation (1) a variable transform $x = x_0 + \alpha^m X$, $y^{(j)} = y_0^j + \alpha^m v^j$ for $j = \overline{0, n-1}$ and obtain a differential system

$$\begin{cases} (v^j)' = y_0^{j+1} + O(\alpha), \; j = \overline{0..n-2} \\ (v^{n-1})' = \alpha^{g_k}\left[\left(\sum_{j=0}^{n-1}\left(\dfrac{\partial P_k}{\partial y^{(j)}}(\lambda)v^j\right) + \dfrac{\partial P_k}{\partial x}(\lambda)X\right)^{g_k/m} H(\lambda) + O(\alpha)\right]. \end{cases} \tag{3}$$

System (3) admits solution of the form

$$\begin{cases} v^j = y_0^{j+1}x + O(\alpha), \; j = \overline{0, n-2}, \\ v^{n-1} = \alpha^{g_k}\left[\gamma^{g_k/m}\dfrac{m}{g_k+m}X^{g_k/m+1}H(\lambda) + O(\alpha)\right] \end{cases},$$

where $\gamma = \sum_{j=0}^{n-2} \frac{\partial P_k}{\partial y^{(j)}}(\lambda) y_0^{j+1} + \frac{\partial P_k}{\partial x}(\lambda)$. If $\gamma \neq 0$ then this solution admits critical singularities in a neighborhood of zero for sufficiently close to zero nonzero α, so the initial Equation (1) in this case admits critical singularities in a neighborhood of x_0, i.e., admits movable critical singularities.

Since k and λ are arbitrary, we have the following condition:

$$\sum_{j=0}^{n-2} \frac{\partial P_k}{\partial a_j}(a^{n-1}, a^{n-2}, \ldots, a^0, x)a^{j+1} + \frac{\partial P_k}{\partial x}(a^{n-1}, a^{n-2}, \ldots, a^0, x) = 0$$

for every k such that g_k/m is not integer and for all $a^{n-1}, a^{n-2}, \ldots, a^0, x$ such that $P_k(a^{n-1}, a^{n-2}, \ldots, a^0, x) = 0$. Since P_k is an irreducible polynomial in $a^{n-1}, a^{n-2}, \ldots, a^0$ and its degree in all variables (determined as a maximum degree of its terms, while the degree of each term is a sum of its degrees in all variable) is not lower than the degree of the above condition's left side, the following condition implies:

$$\sum_{j=0}^{n-2} \frac{\partial P_k}{\partial a}(a^{n-1}, a^{n-2}, \ldots, a^0, x) a^{j+1} + \frac{\partial P_k}{\partial x}(a^{n-1}, a^{n-2}, \ldots, a^0, x)$$
$$= \theta(x) P_k(a^{n-1}, a^{n-2}, \ldots, a^0, x) \tag{4}$$

for all $a^{n-1}, a^{n-2}, \ldots, a^0, x$, where $\theta(x)$ is a locally analytic function in x. The linear partial differential Equation (4) in P_k has a general solution of the form

$$P_k = e^{\left(-\int \theta(x)\,dx\right)} H\left(l_{n-1}, l_{n-2}, \ldots, l_0\right), \tag{5}$$

where H is an arbitrary analytic function in all variables and, here and further, $l_j(v^{n-1}, v^{n-2}, \ldots, v^0, z) = \sum_{i=j}^{n-1} \frac{(n-1-j)!}{(i-j)!}(-z)^{i-j} v^i$. Since P_k is an irreducible polynomial in $a^{n-1}, a^{n-2}, \ldots, a^0$, the function H should also be an irreducible polynomial in all variables.

Now consider the numbers g_j and p_j. The first obvious condition is

$$\sum_{j=1}^{s} g_j p_j = d < 2m, \tag{6}$$

Consider a number g_k for arbitrary k, such that $g_k < m$. In this case the polynomial P_k satisfies the condition (4).

Consider a function $h(a^{n-2}, a^{n-3}, \ldots, a^0, x)$, such that $P_k(h(a^{n-2}, a^{n-3}, \ldots, a^0, x), a^{n-2}, a^{n-3}, \ldots, a^0, x) \equiv 0$. Let $H(y^{(n-1)}, y^{(n-2)}, \ldots, y, x) = \sqrt[m]{R(y^{(n-1)}, y^{(n-2)}, \ldots, y, x)/(y^{(n-1)} - h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x))^{g_k}}$ and consider an arbitrary point $\lambda = (a^{n-2}, a^{n-3}, \ldots, a^0, x_0)$, such that $H(h(\lambda), \lambda) \neq 0$. First, introduce a transformation $y^{(n-1)} = h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x) + v$ and obtain an equation

$$v' = v^{g_k/m} H(h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x) + v, y^{(n-2)}, \ldots, y, x)$$
$$+ \frac{dh(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)}{dx}.$$

Note, that with respect to (4) one can obtain the equality $\frac{dh(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)}{dx} = v\frac{\partial h}{\partial y^{(n-2)}}(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)$. Introduce a variable transform $x = x_0 + \alpha^{m-g_k} X$, $y^{(j)} = a_0^j + \alpha^{m-g_k} v^j$, $j = \overline{0..n-2}$, $v = \alpha^m u$ and, with respect to above equality, obtain a reduced equation

$$u' = u^{g_k/m} H(h(\lambda), \lambda).$$

This equation admits movable critical singularities whenever $m - g_k$ does not divide m.

We proved the following.

LEMMA 1. *If g_k/m is not integer, then $P_k(y^{(n-1)}, y^{(n-2)}, \ldots, y, x) = \theta_k(x) \times H_k(l_{n-1}(y^{(n-1)}, y^{(n-2)}, \ldots, y, x), \ldots\ldots\ldots, l_0(y^{(n-1)}, y^{(n-2)}, \ldots, y, x))$, where H_k is a polynomial in all variables and θ_k is a locally analytic function in x. Moreover, if $g_k < m$ then $g_k = m(1 - 1/\nu_k)$, where ν_k is a natural number.*

Note that Lemma 1 may also be obtained as a consequence of theorem 2 from [13].

From the condition (6) and Lemma 1 we have the following unequlity:

$$m/2 \le g_k < 2m \tag{7}$$

for all k.

Without loss of generality assume that $g_1 \ge g_2 \ge \cdots \ge g_s$. From (6) and (7) one can see that $m > g_2 \ge \cdots \ge g_s \ge m/2$ and $s \le 3$. Also, according to Lemma 1, one can determine the structure of all P_j except of P_1 in case of $g_1 = m$.

3. The case $g_1 = m$ in Equations (1–2)

In the present section we consider a case $g_1 = m$. According to Lemma 1 and (6) for this case we have $s = 2$, $g_2 = m(1 - 1/\nu)$, where ν is an arbitrary natural divisor of m greater than 1, and $p_1 = p_2 = 1$. According to condition (7), Equation (1) may be rewritten in the following form

$$y^{(n)} = L\left(l_{n-1}\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big), \ldots, l_0\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big)\right)^{1-1/\nu} \cdot \left(\sqrt[m]{R_1\big(y^{(n-2)}, y^{(n-3)}, \ldots, y, x\big)}\, y^{(n-1)} + \sqrt[m]{R_2\big(y^{(n-2)}, y^{(n-3)}, \ldots, y, x\big)}\right), \tag{8}$$

where L is a linear form in all variables with constant coefficients, R_1 and R_2 are rational functions in $y, y', \ldots, y^{(n-2)}$ with analytic in x coefficients, while $R_1 \not\equiv 0$.

First show that R_1 and R_2 should be polynomials in $y^{(n-2)}$. Let R_1 or R_2 have a pole along a curve $y^{(n-2)} = h(y^{(n-3)}, y^{(n-4)}, \ldots, y, x)$, where h is analytic in a neighborhood of the certain point $\lambda = (y_0^{n-3}, y_0^{n-4}, \ldots, y_0, x_0)$. Assume $R_i(y^{(n-2)}, y^{(n-3)}, \ldots, y, x) = \tilde{R}_i(y^{(n-2)}, y^{(n-3)}, \ldots, y, x)(y^{(n-2)} - h(y^{(n-2)}, y^{(n-3)}, \ldots, y, x))^{-q_i}$, $i = 1, 2$, where functions $\tilde{R}_i$ are analytic and nonzero in a neighborhood of a point $(h(\lambda), \lambda)$, and q_1, q_2 are integer numbers, provided that at least one of them is positive.

Consider $q = \max\{q_1, q_2\}$. If $q \le m$ then introduce a transformation $x = x_0 + \alpha^{m\nu}X$, $y^{(j)} = y_0^j + \alpha^{m\nu}v^j$, $j = \overline{0..n-3}$, $y^{(n-2)} = h(\lambda) + \alpha^m u$ and obtain a differential system

$$\begin{cases} v' = h(\lambda) + O(\alpha), \\ u'' = (u' + O(\alpha))^{1-1/\nu}(\alpha^{(2-1/\nu)m-q_1}(\gamma^1 u' + O(\alpha))(u + O(\alpha))^{-q_1/m} \\ \qquad + \alpha^{m(\nu+1-1/\nu)-q_2}(\gamma^2 + O(\alpha))(u + O(\alpha))^{-q_2/m}), \end{cases}$$

where $\gamma^1, \gamma^2 \ne 0$. The system admits a solution of the form

$$\begin{cases} v = h(\lambda)(X+C) + O(\alpha), \\ u = X + C + \alpha^{(2-1/\nu)m-q_1}(\gamma^1 + O(\alpha))\left(\iint (X+C)^{-q_1/m} dX\,dX + O(\alpha)\right) \\ \quad + \alpha^{m(\nu+1-1/\nu)-q_2}(\gamma^2 + O(\alpha))\left(\iint (X+C)^{-q_2/m} dX\,dX + O(\alpha)\right). \end{cases}$$

One can see that this solution admits critical (algebraic or logarithmic) singularity in a sufficiently small neighborhood of $x = -C$ for nonzero α sufficiently close to zero. So the initial Equation (8) in the considered case admits movable critical singularities.

Now let $\max\{q_1, q_2\} > m$. One can see that there exists such a rational number $t \in (0, 1)$, that applying a transformation $x = x_0 + \alpha X$, $y^{(j)} = y_0^j + \alpha v^j$, $j = \overline{0..n-3}$, $y^{(n-2)} = h(\lambda) + \alpha^t u$, we obtain a reduced for $\alpha = 0$ differential equation

$$u'' = (u')^{1-1/\nu}(\gamma^1 u' u^{-q_1/m} + \gamma^2 u^{-q_2/m}), \tag{9}$$

where at least one of γ^1, γ^2 is nonzero, while $\min\{(1 - 1/\nu - q_1/m)t + 1/\nu, -(1/\nu + q_2/m)t + 1 + 1/\nu\} = 0$ and $\gamma^1 = 0$ if $(1 - 1/\nu - q_1/m)t + 1/\nu > 0$ and $\gamma^2 = 0$ if $-(1/\nu + q_2/m)t + 1 + 1/\nu > 0$. Equation (9) always admits a solution $u = k(X + C)^t$, where k is a nonzero complex number, C is an arbitrary complex constant. This solution admits critical algebraic singularity at $X = -C$, consequently the initial Equation (8) also admits movable critical singularities in the considered case.

We proved that R_1 and R_2 should be polynomial in $y^{(n-2)}$. Let $g_1 = \deg_{y^{(n-2)}} R_1$, $g_2 = \deg_{y^{(n-2)}} R_2$. In the case $\max\{g_1/m, g_2/m - 2\} > 2/\nu - 1$, then one can see that there exists such a rational number $t \in (0, 1)$, that applying a transformation $x = x_0 + \alpha X$, $y^{(j)} = y_0^j + \alpha v^j$, $j = \overline{0..n-4}$, $y^{(n-3)} = y_0^{n-3} + \alpha^{1-t} Y$, $y^{(n-2)} = \alpha^{-t} u$, where $x_0, y_0^0, y_0^1, \ldots, y_0^{n-3}$ are arbitrary points in a complex plain, we obtain a reduced for $\alpha = 0$ differential equation

$$u'' = \gamma^1 (u')^{2-1/\nu} u^{g_1/m} + \gamma^2 (u')^{1-1/\nu} u^{g_2/m}, \tag{10}$$

where at least one of γ^1, γ^2 is nonzero, while $\min\{(1/\nu - 1 - g_1/m)t + 1/\nu, (1/\nu - g_2/m)t + 1 + 1/\nu\} = 0$ and $\gamma^1 = 0$ if $(1/\nu - 1 - g_1/m)t + 1/\nu > 0$ and $\gamma^2 = 0$ if $(1/\nu - g_2/m)t + 1 + 1/\nu > 0$. Equation (10) admits solution of the form $u = k(X + C)^{-t}$, where k is a nonzero complex number, C is an arbitrary complex constant. This solution admits critical algebraic singularity at $X = -C$, consequently the initial Equation (8) also admits movable critical singularities in the considered case.

So the only possible case is $\max\{g_1/m, g_2/m - 2\} \le 2/\nu - 1$, which is possible only if $\nu = 2$, $g_1 = 0$, $g_2 \le 2m$. In this case, apply a transformation $x = x_0 + \alpha^m X$, $y^{(j)} = y_0^j + \alpha v^j$, $j = \overline{0..n-4}$, $y^{(n-3)} = u$ with arbitrary x_0, $y_0^0, y_0^1, \ldots, y_0^{n-4}$. For $\alpha = 0$ obtain a reduced equation

$$u''' = \sqrt[m]{r_1(u)}(u'')^{3/2} + \sqrt[m]{r_2(u)}(u'')^{1/2}(u')^2, \tag{11}$$

where r_1, r_2 are rational functions in u. This equation has been considered in [4] and presence of movable critical singularities has been proved for it.

Consequently, we proved that Equation (8) always admits movable critical singularities, i.e., there are no Equations (1) with $g_1 = m$ free of movable critical singularities.

4. The case $g_1 \neq m$ in Equations (1–2)

Now consider a case $g_1 \neq m$. In this case the Equation (1) may be rewritten in the following form

$$y^{(n)} = \prod_{j=1}^{s} P_j\left(l_{n-1}\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big), \ldots, l_0\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big)\right)^{g_j/m} \times \sqrt[m]{R_1\big(y^{(n-2)}, y^{(n-3)}, \ldots, y, x\big)}, \tag{12}$$

where $s \leq 3$, P_j are polynomials in all variables with constant coefficients of degree $p_j, r = \sum_{j=1}^{s} p_j g_j/m < 2$, and if $g_j < m$ then $g_j/m = 1 - 1/\nu_j$, where $\nu_j \in N, \nu_j > 1$; R_1 is a rational function in y and its derivatives with coefficients locally analytic in x.

First consider a case when R_1 essentially depends on y or its derivatives.

Let τ be the minimal integer number not less than -1, such that R_1 is a polynomial in $y^{(j)}$ for all $j \geq \tau + 1$. If R_1 is not a polynomial even in $y^{(n-2)}$ let $\tau = n - 2$. If R_1 is polynomial in y and all of its derivatives let $\tau = -1$. Then Equation (12) may be rewritten in the following form

$$y^{(n)} = \prod_{j=1}^{s} P_j\left(l_{n-1}\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big), \ldots, l_0\big(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\big)\right)^{g_j/m} \cdot \sqrt[m]{\sum_{\chi \in S} \big(y^{(n-2)}\big)^{\chi_{k-2}} \big(y^{(n-3)}\big)^{\chi_{k-3}} \ldots \big(y^{(\tau+1)}\big)^{\chi_1} f_\chi\big(y^{(\tau)}, y^{(\tau-1)}, \ldots, y, x\big)}, \tag{13}$$

where $k = n - \tau$, S is a certain set of points $\chi = (\chi_1, \chi_2, \ldots, \chi_{k-2})$ in Z^{k-2}, f_χ are not identically equal to zero rational functions in y and its derivatives with analytic in x coefficients.

For $j = \overline{0, k-2}$ let

$$|\chi|_j = \sum_{i=j+1}^{k-2} \chi_i, \;\; |\chi| = |\chi|_0, \;\; \nu_j(\chi) = \sum_{i=j+1}^{k-2} (i - j)\chi_i,$$

$$\theta_j = ((r-1)(k-j) - r)m + \max_{\chi \in S}\{\nu_j(\chi)\}, \; f_\chi^j(y^{(\tau+j)}, y^{(\tau+j-1)} \ldots y, x) = \big(y^{(\tau+j)}\big)^{\chi_j} \big(y^{(\tau+j-1)}\big)^{\chi_{j-1}} \big(y^{(\tau+1)}\big)^{\chi_1} f_\chi\big(y^{(\tau)}, y^{(\tau-1)} \ldots, y, x\big).$$

One can see that $\nu_{j+1}(\chi) = \nu_j(\chi) - |\chi|_j$ and $\nu_{k-2}(\chi) = |\chi|_{k-2} = 0$, $\theta_{k-2} = r - 2 < 0$.

There are three possibilities: (1) $\theta_0 \geq 0$; (2) $\theta_0 < 0$ and $\tau \geq 0$; (3) $\tau = -1$ and $\theta_0 < 0$.

4.1. Subcase 1

Let $\hat{\tau} = \max_{0 \leq j \leq k-2,\ \theta_j \geq 0} j$, $\hat{k} = k - \hat{\tau}$. One can see that $0 \leq \hat{\tau} < k - 2$ and $\theta_{\hat{\tau}+1} < 0$. Again consider two cases: (1a) $\theta_{\hat{\tau}} > 0$; and (1b) $\theta_{\hat{\tau}} = 0$.

(1a). In this case apply to Equation (12) a variable transformation $x = x_0 + \alpha^M X$, $y^{(j)} = y_0^j + \alpha^M y^j$, $j = \overline{0, n - \hat{k} - 1}$, $y^{(n-\hat{k})} = y_0^{n-\hat{k}} + \alpha^{tM} u$, where x_0, y_0^j are some complex constants, t is a positive rational number, M is a natural number, such that tM is integer, and obtain a system

$$\begin{cases} (y^j)' = y_0^{j+1} + O(\alpha),\ j = \overline{0, n - \hat{k} - 1} \\ u^{(\hat{k})} = ((u^{(\hat{k}-1)})^r + O(\alpha)) \\ \qquad \times \sqrt[m]{\sum_{\chi \in S} \alpha^{M\left[((r-1)m + |\chi|_{\hat{\tau}})t + (\hat{k} - (\hat{k}-1)r)m - \nu_{\hat{\tau}}(\chi)\right]} (u^{(\hat{k}-2)})^{\chi_{k-2}} (u^{(\hat{k}-3)})^{\chi_{k-3}} \ldots (u')^{\chi_{\hat{\tau}+1}} (c_\chi + O(\alpha))}, \end{cases} \tag{14}$$

where $c_\chi = f_\chi^{\hat{\tau}}(y_0^{n-\hat{k}}, y_0^{n-\hat{k}-1}, \ldots, y_0^0, x_0)$. Consider a function $\varphi(t) = \min_{\chi \in S}\{((r-1)m + |\chi|_{\hat{\tau}})t + (\hat{k} - (\hat{k}-1)r)m - \nu_{\hat{\tau}}(\chi)\}$. It is continuous on the segment [0, 1], while $\varphi(0) = -\theta_{\hat{\tau}} < 0$, $\varphi(1) = -\theta_{\hat{\tau}+1} > 0$. Consequently function φ admits at least one root $t = t_0$ on the interval (0, 1). If $t = t_0$, obtain the reduced for $\alpha = 0$ equation of the following form

$$u^{(\hat{k})} = (u^{(\hat{k}-1)})^r \sqrt[m]{\sum_{\chi \in S_0} c_\chi \cdot (u^{(\hat{k}-2)})^{\chi_{k-2}} (u^{(\hat{k}-3)})^{\chi_{k-3}} \ldots (u')^{\chi_{\hat{\tau}+1}}},$$

where $S_0 = \{\chi \in S | ((r-1)m + |\chi|_{\hat{\tau}})t_0 + (\hat{k} - (\hat{k}-1)r)m - \nu_{\hat{\tau}}(\chi) = 0\} \neq \emptyset$.

According to the following Lemma 2, this equation admits movable critical singularities, consequently, Equation (12) in the considered case also admits them.

LEMMA 2. *Consider the equation*

$$y^{(n)} = \sum_{\chi \in S} c_\chi \cdot \left(y^{(n-2)}\right)^{\chi_{n-2}} \left(y^{(n-3)}\right)^{\chi_{n-3}} \ldots y^{\chi_0}, \tag{15}$$

where n is a natural number, r is a positive rational number, S is a certain set of points $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-2})$ in Q^{n-1}, c_χ are nonzero complex numbers. Let $\nu(\chi) = \sum_{j=1}^{n-2} j\chi_j$, $|\chi| = \sum_{j=0}^{n-2} \chi_j$ and suppose that $|\chi| > 1$ for all χ. Also suppose that there exists a rational number t, such that for every $\chi \in S$ the equality $\nu(\chi) - n = t(|\chi| - 1)$ holds.

Then Equation (15) either admits movable critical singularities or admits a family of solutions of the form $y = q(x + C)^t$, where q is a nonzero complex number and C is an arbitrary complex constant (if t is not integer Equation (15) admits movable critical singularities anyway).

Proof of the Lemma 2: Look for a family of solutions of the form $y = q(x + C)^t$ for the Equation (15). These solutions satisfy Equation (15) if and only if q satisfies an equation

$$\zeta(t, n)q = \sum_{\chi \in S} c_\chi q^{|\chi|}\zeta(t, n-2)^{\chi_{n-2}}\zeta(t, n-3)^{\chi_{n-3}} \dots \zeta(t, 0)^{\chi_0}, \tag{16}$$

where $\zeta(t, j)$ denotes an expression $t(t - 1) \cdot \ldots \cdot (t - j + 1)$. So we only have to consider a case, when Equation (16) admits no nonzero roots. Note that in, this case, the function $y = q(x + C)^t$ should satisfy the equations $0 = \sum_{\chi \in S, |\chi|=d} c_\chi (y^{(n-2)})^{\chi_{n-2}}(y^{(n-3)})^{\chi_{n-3}} \dots y^{\chi_0}$ for all integer d and complex q.

Apply to Equation (15) a transform $x = -C + e^{X/\alpha}$, $y = \alpha^{-r}\ (x + C)^t u$, where r is a certain positive rational, and obtain an equation

$$\alpha^{M/T} u^{(n)} = F\big(u^{(n-1)}, u^{(n-2)}, \dots, u, \alpha^{1/T}\big), \tag{17}$$

where T *and* M are certain natural numbers, F is a function, analytic in all variables in a neighborhood of a point $(0, 0, \dots, 0, 1, 0)$, while $F(0_{C^{n-1}}, u, 0) \equiv 0$ and $F(0_{C^{n-1}}, u, \alpha) \not\equiv 0$.

From the theorem 1 and remark 2 from [14] one can see that for any natural K there exists a solution $u = 1 + \sum_{j=1}^{KB-1} \varphi_j(X)\alpha^{j/B} + \alpha^K \phi(X, \alpha)$ of Equation (17) for positive real α, where B is a certain natural number, functions φ_j are analytic in a certain complex domain U, ϕ is analytic in X for $X \in U$ and $\phi(X, \alpha) \to 0$ for $\alpha \to 0$ uniformly on U. Without loss of generality the domain U may be considered a neighborhood of zero, because Equation (17) is autonomous.

So for the initial Equation (15) we have a solution

$$y = \alpha^{-r}(x + C)^t \left(1 + \sum_{j=1}^{KB-1} \varphi_j(\alpha \ln(x + C))\alpha^{j/B} + \alpha^K \phi(\alpha \ln(x + C), \alpha)\right)$$
$$= \alpha^{-r}(x + C)^t \left(1 + \sum_{j=1}^{KB-1} \tilde{\varphi}_j(\ln(x + C))\alpha^{j/B} + \alpha^K \tilde{\phi}(x, \alpha)\right)$$

where $\tilde{\varphi}_j$ are polynomials with constant coefficients; $\tilde{\phi}$ is analytic in x for $x \in V = \{z\colon 0 < r_1 \le |z + C| \le r_2\}$, where r_1 is a sufficiently small positive real, r_2 is a sufficiently great positive real; $\tilde{\phi}(x, \alpha) \to 0$ for $\alpha \to 0$ uniformly on V. Because $F(0_{C^{n-1}}, u, \alpha) \not\equiv 0$ one can see that for sufficiently great K not all of the polynomials $\tilde{\varphi}_j$ could be constant. Consequently the given solution possesses the logarithmic critical singularity near $x = -C$ for sufficiently small positive real α. The proof of Lemma 2 is now complete. ■

(1b). Apply a variable transform $x = x_0 + \alpha X, y^{(j)} = y_0^j + \alpha y^j, j = \overline{0, n - \hat{k} - 1}, y^{(n-\hat{k})} = u$, where x_0, y_0^j are some complex constants, to Equation (13) and obtain a reduced for $\alpha = 0$ equation of the form

$$u^{(\hat{k})} = \left(u^{(\hat{k}-1)}\right)^r \sqrt[m]{\sum_{\chi \in S_0} \left(u^{(\hat{k}-2)}\right)^{\chi_{\hat{k}-2}} \left(u^{(\hat{k}-3)}\right)^{\chi_{\hat{k}-3}} \dots (u')^{\chi_1} \hat{f}_\chi(u)}, \qquad (18)$$

where $\hat{f}_\chi(u)$ are not identically equal to zero rational functions and S_0 is a certain set of points $\chi = (\chi_1, \chi_2, \dots, \chi_{\hat{k}-2})$ in $Z^{\hat{k}-2}$, such that $\nu(\chi) = \sum_{j=1}^{\hat{k}-2} j\chi_j = (\hat{k} - (\hat{k} - 1)r)m$. Denote $|\chi| = \sum_{j=1}^{\hat{k}-2} \chi_j$.

Consider Equation (18). First suppose that all of its coefficients $\hat{f}_\chi(u)$ are polynomials in u. Let q_χ be the degree of $\hat{f}_\chi(u)$ and let c_χ be its major coefficient. Let $t = \max_{\chi \in S_0} \frac{q_\chi}{(r-1)m + |\chi|}$. One can see that $t \geq 0$, because $|\chi| + (r - 1)m > \nu(\chi)/\hat{k} + (r - 1)m - rm/\hat{k} = 0$ for all $\chi \in S_0$.

Apply to Equation (18) a transform $u = \alpha^{-M} v$, $u' = \alpha^{tM}\omega$, where M is a natural number such that Mt is integer, and obtain the following system

$$\begin{cases} \omega^{(\hat{k}-1)} = (\omega^{(\hat{k}-2)})^r \\ \qquad \times \sqrt[m]{\sum_{\chi \in S_0} (\alpha^{M(t((r-1)m+|\chi|_{\hat{\tau}})-q_\chi)} (\omega^{(\hat{k}-3)})^{\chi_{k-2}} (\omega^{(\hat{k}-4)})^{\chi_{k-3}} \dots \omega^{\chi_{\hat{\tau}+1}} (c_\chi v^{q_\chi} + O(\alpha))}, \\ v' = \alpha^{(t+1)M}\omega, \end{cases} \qquad (19)$$

Consider a reduced for $\alpha = 0$ equation

$$\omega^{(\hat{k}-1)} = (\omega^{(\hat{k}-2)})^r \sqrt[m]{\sum_{\chi \in S_1} c_\chi \left(\omega^{(\hat{k}-3)}\right)^{\chi_{k-2}} \left(\omega^{(\hat{k}-4)}\right)^{\chi_{k-3}} \dots \omega^{\chi_{\hat{\tau}+1}} v_0^{q_\chi}}, \qquad (20)$$

where $S_1 = \{\chi \in S_0 \,|\, q_\chi = t((r - 1)m + |\chi|)\}$, v_0 is an arbitrary complex constant. If $\omega_0(x)$ is a solution of Equation (20), then system (19) admits a solution of the form

$$\omega = \omega_0(x) + O(\alpha), \;\; v = v_0 + \left(\int \omega_0(x)\, dx + O(\alpha)\right) \alpha^{(t+1)M}. \qquad (21)$$

But, according to Lemma 2, the solution ω_0 of Equation (20) admits either movable critical singularities or single poles. In both cases the solution (21) admits movable critical singularities.

Consequently not all of the coefficients $\hat{f}_\chi(u)$ should be polynomials. Obviously it means that $\hat{\tau} = 0$, $S_0 = \{\chi \in S : \nu_0(\chi) = (k - (k - 1)r)m\}$. Let $u = u_0$ be a pole of at least one of the functions $\hat{f}_\chi(u)$. For all χ let

$\hat{f}_\chi(u) = (u - u_0)^{-q_\chi} \tilde{f}_\chi$, where q_χ is an integer and $\tilde{f}_\chi$ is a function, analytic in $u = u_0$, such that $\tilde{f}_\chi(u_0) = c_\chi \neq 0$.

Suppose that there exists a $\chi \in S_0$, such that $q_\chi > (r-1)m + |\chi|$. Then apply a transform $u = u_0 + \alpha^M v$, $u' = \alpha^{tM}\omega$, where $t = \max_{\chi \in S_0} \frac{q_\chi}{(r-1)m+|\chi|} > 1$ and M is a natural number such that tM is integer, and obtain a system

$$\begin{cases} \omega^{(k-1)} = (\omega^{(k-2)})^r \\ \qquad \times \sqrt[m]{\sum_{\chi \in S_0} \alpha^{M(t((r-1)m+|\chi|)-q_\chi)} (\omega^{(k-3)})^{\chi_{k-2}} (\omega^{(k-4)})^{\chi_{k-3}} \cdot \ldots \cdot \omega^{\chi_1} (c_\chi v^{-q_\chi} + O(\alpha))}, \\ v' = \alpha^{(t-1)M}\omega, \end{cases}$$

One can show that this system admits movable critical singular points just in the same way as it has been done for the system (19).

Consequently, if Equation (18) is free of movable critical singularities, then for all χ the inequality $q_\chi \leq (r-1)m + |\chi|$ should hold. Apply a transform $u = u_0 + \alpha^m v$ and obtain an equation

$$v^{(k)} = (v^{(k-1)})^r \times \sqrt[m]{\sum_{\chi \in S_0} \alpha^{(r-1)m+|\chi|-q_\chi} (v^{(k-2)})^{\chi_{k-2}} (v^{(k-3)})^{\chi_{k-3}} \cdot \ldots \cdot (v')^{\chi_1} (c_\chi v^{-q_\chi} + O(\alpha))}. \tag{22}$$

If for all χ we have $q_\chi < (r-1)m + |\chi| < mk$ then Equation (22) possesses a solution of the form

$$v = A(X) + \iint \ldots \int (A^{(k-1)}(X))^r \cdot \sqrt[m]{\sum_{\chi \in S_0} \alpha^{(r-1)m+|\chi|-q_\chi} (A^{(k-2)}(X))^{\chi_{k-2}} (A^{(k-3)}(X))^{\chi_{k-3}} \cdot \ldots \cdot (A'(X))^{\chi_1} (c_\chi (A(X))^{-q_\chi} + O(\alpha))}\, dX^k,$$

where A is an arbitrary polynomial of degree not higher than $k-1$. This solution, as one can see, admits movable critical (algebraic or logarithmic) singularities.

Otherwise, if there exists χ such, that $q_\chi = (r-1)m + |\chi|$, consider the reduced for $\alpha = 0$ equation

$$v^{(k)} = (v^{(k-1)})^r \sqrt[m]{\sum_{\chi \in S_1} c_\chi \left(v^{(k-2)}\right)^{\chi_{k-2}} \left(v^{(k-3)}\right)^{\chi_{k-3}} \cdot \ldots \cdot (v')^{\chi_1} v^{-q_\chi}}, \tag{23}$$

where $S_1 = \{\chi \in S_0 : q_\chi = (r-1)m + |\chi|\}$.

If $r > 1$ then the only possible case for Equation (23) is

$$v^{(k)} = c\left(v^{(k-1)}\right)^r / v^{r-1}, \tag{24}$$

while $r = k/(k-1) < 2$ (so $k \geq 3$) and c is a nonzero complex coefficient. But according to theorem 1 of [13] the Equation (24) also admits movable critical singularities.

If $r = 1$ then the only possible case is $s = 2$, $r_1 = r_2 = 1/2$ and Equation (23) is

$$v^{(k)} = cv^{(k-1)}v'/v, \tag{25}$$

where c is a nonzero complex number. This equation for $c \neq 1$ admits a family of solutions $v = C_2(x + C_1)^t$, where C_1, C_2 are arbitrary complex constants, and $t = (k-1)/(1-c)$. So c should be at least rational number. One can also consider c to be negative (in the opposite case it is sufficient to consider another branch of right-hand side of Equation (13) and obtain Equation (25) with coefficient $-c$). The Equation (25) admits first integral of the form $v^{(k-1)} = Cv^c$, where C is an arbitrary complex constant. According to the theorem 3 of [13], this equation admits movable critical singularities and so does the initial Equation (25).

Let $r < 1$. In this case $r = 1 - 1/\mu$, while μ is an integer greater than 1. Equation (23) admits a family of solutions of the form $v = C_1(X + C_2)^t$, where C_1, C_2 are arbitrary complex constants and t is a root of a so called determining equation

$$(t-k+1)^{\mu}\zeta(t,k-1) = \left(\sqrt[m]{\sum_{\chi\in S_1} c_\chi(\zeta(t,k-2))^{\chi_{k-2}}(\zeta(t,k-3))^{\chi_{k-3}}\cdot\ldots\cdot(\zeta(t,1))^{\chi_1}}\right)^{\mu}, \tag{26}$$

where, here and further, $\zeta(t,j)$ denotes an expression $t(t-1)\cdot\ldots\cdot(t-j+1)$. So if Equation (23) is free of movable critical singularities then all of the roots of Equation (26) should be integer. According to theorem 1 of [13] it is also necessary that the expression $\sqrt[m]{\sum_{\chi\in S_1} c_\chi(v^{(k-2)})^{\chi_{k-2}}(v^{(k-3)})^{\chi_{k-3}}\cdot\ldots\cdot(v')^{\chi_1}v^{-q_\chi}}$ should be polynomial in the derivatives of v and so the expression $\sqrt[m]{\sum_{\chi\in S_1} c_\chi(\zeta(t,k-2))^{\chi_{k-2}}(\zeta(t,k-3))^{\chi_{k-3}}\cdot\ldots\cdot(\zeta(t,1))^{\chi_1}}$ should be polynomial in t. Thus the right-hand side of Equation (26) is a polynomial divided by t^{μ}, while the left-hand side is divided by t and not divided by t^2. Consequently Equation (26) possesses at least one nonzero root $t = t_0$.

Let $v = (X+C)^{t_0} + \alpha u$. Obtain a reduced for $\alpha = 0$ Euler linear differential equation

$$u^{(k)} = \frac{r(t_0-k+1)}{X+C}u^{(k-1)} + \sum_{j=0}^{k-2}\frac{a_j}{(X+C)^{k-j}}u^{(j)}, \tag{27}$$

where a_j are complex numbers. Equation (27) admits solutions of the form $u = (X+C)^{\rho}$, where ρ is an arbitrary root of an equation

$$\zeta(\rho,n) = r(t_0-n+1)\zeta(\rho,n-1) + \sum_{j=0}^{n-2} a_j\zeta(\rho,j), \tag{28}$$

where $\zeta(\rho, j)$ denotes an expression $\rho(\rho - 1) \cdot \ldots \cdot (\rho - j + 1)$. If Equation (23) is free of movable critical singularities, then all the roots of Equation (28) should be integer. Moreover, Equation (28) should not admit multiple roots because in the opposite case, Equation (27) admits solutions with logarithmic singularity in $X = -C$.

One can see that Equation (28) always admits a root $\rho = t_0 - 1$. Denote other $n - 1$ roots $\rho_1, \rho_2, \ldots, \rho_{n-1}$, while $\rho_1 < \rho_2 < \cdots < \rho_{n-1}$.

First consider a case when $\chi_{n-2} = 0$ for all $\chi \in S_1$. In this case $a_{n-2} = 0$. Then from Equation (28) one can see that the following conditions hold

$$\sum_{j=1}^{n-1} \rho_j = \sum_{j=1}^{n-1}(j-1) + (1-r)(n-t_0) + r, \tag{29}$$

$$\sum_{j=1}^{n-1} \rho_j^2 = \sum_{j=1}^{n-1}(j-1)^2 + ((1-r)(n-t_0)+r)((n-1)(1-r)+t_0(r+1)-1). \tag{30}$$

We will need the following.

LEMMA 3. *Let $a_1, a_2, \ldots, a_n$ be pairwise different integer numbers and $\sum_{j=1}^n a_j = \sum_{j=1}^n (j-1) + l$, where l is an integer number. Then $\sum_{j=1}^n a_j^2 \geq \sum_{j=1}^n (j-1)^2 + nl$.*

Proof of Lemma 3: Without loss of generality one can assume that $a_1 < a_2 < \cdots < a_n$. Let $a_j = j - 1 + \delta_j$. Then $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_n$ and $\sum_{j=1}^n \delta_j = l$. Note that for all $j \leq n/2$ the following inequality holds:

$$\begin{aligned}
&\delta_j(j-1) + \delta_{n+1-j}(n-j) \\
&\quad = (\delta_j + \delta_{n+1-j})(n-1)/2 \\
&\qquad + (\delta_{n+1-j} - \delta_j)(n-2j+1) \geq (\delta_j + \delta_{n+1-j})(n-1)/2.
\end{aligned}$$

Also note that $\sum_{j=1}^n \delta_j^2 \geq (\sum_{j=1}^n |\delta_j|)^2/n_1 \geq \sum_{j=1}^n |\delta_j| \geq \sum_{j=1}^n \delta_j = l$, where n_1 is the number of nonzero δ_j, while, of course, $n_1 \leq \sum_{j=1}^n |\delta_j|$.

Consequently we have the following inequality

$$\begin{aligned}
\sum_{j=1}^n a_j^2 &= \sum_{j=1}^n (j-1)^2 + 2\sum_{j=1}^n (j-1)\delta_j + \sum_{j=1}^n \delta_j^2 \\
&\geq \sum_{j=1}^n (j-1)^2 + (n-1)\sum_{j=1}^n \delta_j + l = \sum_{j=1}^n (j-1)^2 + nl.
\end{aligned}$$

The proof of Lemma 3 is now complete. ■

Now from (29) and Lemma 3 we have the following inequality

$$\sum_{j=1}^{n-1} \rho_j^2 \geq \sum_{j=1}^{n-1} (j-1)^2 + ((1-r)(n-t_0)+r)(n-1) > \sum_{j=1}^{n-1} (j-1)^2$$
$$+ ((1-r)(n-t_0)+r)((n-1)(1-r)+t_0(r+1)-1),$$

which contradicts (30).

Now let the the right-hand side of Equation (23) depends on $v^{(k-2)}$. An expression $\sqrt[m]{\sum_{\chi\in S_1} c_\chi (v^{(k-2)})^{\chi_{k-2}} (v^{(k-3)})^{\chi_{k-3}} \cdot \ldots \cdot (v')^{\chi_1} v^{-q_\chi}}$ should be polynomial in the derivatives of v. In this case there exists $\chi \in S_1$ such that $\chi_{k-2} \geq m$ and so $\nu_0(\chi) \geq m(k-2)$. Now from $[k-(k-1)r]m - \nu_0(\chi) = \theta_0 = 0$ we have $k \leq 2/r + 1 \leq 5$. Furthermore, from the conditions $[k-(k-1)r]m - \nu_0(\chi) = 0$ and $q_\chi = (r-1)m + |\chi|$ for all χ, one can see that only the following three cases for Equation (23) are possible:

$$v''' = a(v'')^{1/2} v'^2 / v^{3/2}, \tag{31}$$

$$v^{IV} = (v'''/v)^{2/3} (av'' + bv'^2/v), \tag{32}$$

$$v^{V} = (v^{IV}/v)^{1/2} (av''' + bv''v'/v + c(v')^3/v^2), \tag{33}$$

where a, b, c are arbitrary complex coefficients.

Equation (31) is a particular case of Equation (11), so it admits movable critical singularities.

Equations (32) and (33) also admit movable critical singularities. One can prove it by considering their determining equations together with the corresponding Euler equations for each of the determining equation's roots.

4.2. Subcase 2

Consider the case $\theta_0 < 0, \tau \geq 0$. In this case functions f_χ are not all polynomials in $y^{(\tau)}$, i.e., admit poles in it. Let the curve $y^{(\tau)} = h(y^{(\tau-1)}, y^{(\tau-2)}, \ldots, y, x)$ be one of them (here h is an analytic function in all variables in a neighborhood of some point $\lambda = (y_0^{\tau-1}, y_0^{\tau-2}, \ldots, y_0, x_0) \in C^{\tau+1}$). For all χ let $f_\chi(y^{(\tau)}, y^{(\tau-1)}, \ldots, y, x) = (y^{(\tau)} - h(y^{(\tau-1)}, y^{(\tau-2)}, \ldots, y, x))^{-q_\chi} \tilde{f}_\chi(y^{(\tau)}, y^{(\tau-1)}, \ldots, y, x)$, where $\tilde{f}_\chi$ are functions, analytic and nonzero in a neighborhood of a point $\tilde{\lambda} = (y_0^\tau, \lambda)$, where $y_0^\tau = h(\lambda)$. Let $\theta^* = {}_{\chi\in S} \{[(r-1)(k-1)-r]m + \nu_1(\chi) + q_\chi\}$. Consider two cases: (2a) $\theta^* > 0$; (2b) $\theta^* \leq 0$.

(2a). In this case apply to Equation (13) a transform $x = x_0 + \alpha X$, $y^{(j)} = y_0^j + \alpha^M y^j$, $j = \overline{0, \tau-1}$, $y^{(\tau)} = y_0^\tau + \alpha^{tM} u$ for a certain rational number $t \in$

(0, 1) and natural M such that tM is integer, and obtain a reduced for $\alpha = 0$ equation

$$u^{(k)} = (u^{(k-1)})^r \sqrt[m]{\sum_{\chi \in S} c_\chi \left(u^{(k-2)}\right)^{\chi_{k-2}} \left(u^{(k-3)}\right)^{\chi_{k-3}} \dots (u')^{\chi_1} u^{-q_\chi}},$$

where $S_0 = \{\chi \in S \,|\, ((r-1)m + |\chi|_0 - q_\chi)t + (k-(k-1)r)m - \nu_0(\chi) = 0\} \neq \emptyset$. According to Lemma 2, this equation admits movable critical singularities, consequently, Equation (13) in the considered case also admits them.

(2b). In this case for all χ we have $\nu_0(\chi)$, $q_\chi < km$. Apply to Equation (13) a transform $x = x_0 + \alpha X$, $y^{(j)} = y_0^j + \alpha y^j$, $j = \overline{0, \tau - 1}$, $y^{(\tau)} = u + h(y_0^{\tau-1}, y_0^{\tau-2}, \dots, y_0^0, x_0)$ and obtain a system

$$\begin{cases} (y^j)' = y_0^{j+1} + O(\alpha), \quad j = \overline{0, \tau - 2} \\ (y^{\tau-1}) = u, \\ u^{(k)} = \left(\left(u^{(k-1)}\right)^r + O(\alpha)\right) \\ \qquad \times \sqrt[m]{\sum_{\chi \in S} \alpha^{[(k-(k-1)r)m - \nu_0(\chi)]} (u^{(k-2)})^{\chi_{k-2}} (u^{(k-3)})^{\chi_{k-3}} \dots (u')^{\chi_1} (u + O(\alpha))^{-q_\chi} (c_\chi(u) + O(\alpha))}, \end{cases} \tag{34}$$

where $c_\chi(u) = \tilde{f}_\chi(u, y_0^{\tau-1}, y_0^{\tau-2}, \dots, y_0^0, x_0)$. The system (34) admits a solution of the form

$$u = A(X) + \iint \dots \int \left(\left(A^{(k-1)}(X)\right)^r + O(\alpha)\right)$$
$$\cdot \sqrt[m]{\sum_{\chi \in S} \alpha^{[(k-(k-1)r)m - \nu_0(\chi)]} \left[(A^{(k-2)}(X))^{\chi_{k-2}} (A^{(k-3)}(X))^{\chi_{k-3}} \dots (A'(X))^{\chi_1} (A(X) + O(\alpha))^{-q_\chi} c_\chi(A(X)) + O(\alpha)\right]} dX^k,$$

where A is an arbitrary polynomial of degree not higher than $k-1$ with complex coefficients. One can see that this solution admits movable critical singularities for sufficiently close to zero nonzero α. That is why Equation (13) in the considered case admits movable critical singularities.

4.3. Subcase 3

In this case $\tau = -1$ and $f_\chi = f_\chi(x)$. Apply to Equation (13) a transform $x = x_0 + \alpha X$, $y = \alpha^t v$ and obtain an equation

$$v^{(n)} = (v^{(n-1)})^r$$
$$\times \sqrt[m]{\sum_{\chi \in S} \alpha^{t(|\chi| + m(r-1)) + m(n-(n-1)r) - \nu_1(\chi)} (v^{(n-2)})^{\chi_{n-1}} (v^{(n-3)})^{\chi_{n-2}} \dots v^{\chi_1} (f_\chi(x_0) + O(\alpha))}. \tag{35}$$

Consider a function $\varphi(t) = \min_{\chi \in S} \{t(|\chi| + m(r-1)) + m(n-(n-1)r) - \nu_1(\chi)\}$. It is continuous on $(-\infty, -1]$ and $\varphi(-1) = -\theta_0 > 0$, while $\lim_{t \to -\infty}$

$\varphi(t) = -\infty$. Consequently, function φ admits at least one root $t = t_0 < -1$. For $t = t_0$, obtain from (35) a reduced for $\alpha = 0$ equation of the following form

$$v^{(n)} = \left(v^{(n-1)}\right)^r \sqrt[m]{\sum_{\chi \in S_0} \left(v^{(n-2)}\right)^{\chi_{n-1}} \left(v^{(n-3)}\right)^{\chi_{n-2}} \dots v^{\chi_1} f_\chi(x_0)}, \qquad (36)$$

where $S_0 = \{\chi \in S | t_0(|\chi| + m(r-1)) + m(n-(n-1)r) - \nu_1(\chi) = 0\}$. According to theorem 1 of [13], if Equation (36) is free of movable critical singularities, then the equation's right-hand side should be polynomial in v, $v', \dots, v^{(n-2)}$.

Consequently there exists such $\chi \in S_0$ that $|\chi| \geq m$. For this χ one can obtain $0 \leq t_0(|\chi| + m(r-1)) + m(n-(n-1)r) - \nu_1(\chi) \leq m(1-r)$, so $r \leq 1$.

From Lemma 2, if Equation (36) is free of movable critical singularities it should admit solutions of the form $v = q(X+C)^{t_0}$ where q is a nonzero complex number and C is an arbitrary complex constant. Let $v = q(X+C)^{t_0} + \alpha v$. Obtain a reduced equation for $\alpha = 0$ Euler linear differential equation

$$u^{(n)} = \frac{r(t_0 - n + 1)}{X + C} u^{(n-1)} + \sum_{j=0}^{n-2} \frac{a_j}{(X+C)^{n-j}} u^{(j)}, \qquad (37)$$

where a_j are complex numbers. Equation (37) admits solutions of the form $u = (X+C)^\rho$, where ρ is an arbitrary root of an equation

$$\zeta(\rho, n) = r(t_0 - n + 1)\zeta(\rho, n-1) + \sum_{j=0}^{n-2} a_j \zeta(\rho, j), \qquad (38)$$

where $\zeta(\rho, j)$ denotes an expression $\rho(\rho-1) \cdot \ldots \cdot (\rho - j + 1)$. If Equation (36) is free of movable critical singularities, then all the roots of Equation (38) should be integer. Moreover, Equation (38) should not admit multiple roots because in the opposite case Equation (37) admits solutions with logarithmic singularity at $X = -C$. One can see that Equation (38) always admits a root $\rho = t_0 - 1$. Denote other $n-1$ roots $\rho_1, \rho_2, \dots, \rho_{n-1}$, while $\rho_1 < \rho_2 < \cdots < \rho_{n-1}$.

First consider a case when $\chi_{n-2} = 0$ for all $\chi \in S_0$. In this case $a_{n-2} = 0$. Then from (38) one can see that the following conditions hold

$$\sum_{j=1}^{n-1} \rho_j = \sum_{j=1}^{n-1} (j-1) + (1-r)(n-t_0) + r, \qquad (39)$$

$$\sum_{j=1}^{n-1} \rho_j^2 = \sum_{j=1}^{n-1} (j-1)^2 + ((1-r)(n-t_0) + r)((n-1)(1-r) + t_0(r+1) - 1). \qquad (40)$$

But from condition (39) and Lemma 3 we have

$$\sum_{j=1}^{n-1}\rho_j^2 \geq \sum_{j=1}^{n-1}(j-1)^2 + ((1-r)(n-t_0)+r)(n-1) > \sum_{j=1}^{n-1}(j-1)^2$$
$$+((1-r)(n-t_0)+r)((n-1)(1-r)+t_0(r+1)-1),$$

what contradicts (40).

Now let the the right-hand side of Equation (36) depend on $v^{(k-2)}$, i.e., let there exist such a $\chi \in S_0$ that $\chi_{n-2} \neq 0$. Then, because the right-hand side of (36) should be polynomial in $v, v', \ldots, v^{(n-2)}$, the inequality $\chi_{n-2} \geq m$ should hold. Then for this χ one can obtain the following condition $0 = t_0(|\chi| + m(r-1)) + m(n-(n-1)r) - \nu_1(\chi) \leq -mr + m(n-(n-1)r) - (n-2)m = m(2-nr)$, i.e., $n \leq 2/r \leq 4$.

One can see that only the following case is possible for Equation (36):

$$v^{(IV)} = (v''')^{1/2}(av'' + bv'v + cv^3), \tag{41}$$

where a, b, c are complex coefficients. The corresponding case for the initial Equation (12) is:

$$y^{(IV)} = \left(h(z)y''' - h'(z)y'' + h''(z)y' - h'''(z)y + \psi\right)^{1/2}$$
$$\cdot\left(A(x)y'' + B(x)y'y + C(x)y^3 + D(x)y' + E(x)y^2 + F(x)y + G(x)\right), \tag{42}$$

where A, B, C, D, E, F, G are locally analytic functions in x; h is an arbitrary polynomial or order not higher than 3 with constant coefficients, ψ is an arbitrary constant.

5. Equation (12) for $R_1 = R_1(x)$

Now consider a case, when R_1 does not depend on y and its derivatives, i.e., $R_1 = R_1(x)$ and Equation (12) is of the following form

$$y^{(n)} = f(x)\prod_{j=1}^{s} P_j\left(l_{n-1}\left(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\right), \ldots,\right.$$
$$\left. l_0\left(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\right)\right)^{\beta_j}, \tag{43}$$

where $f(x) = \sqrt[m]{R_1(x)}$ is a function, analytic in some domain, $\beta_j = g_j/m$. Apply a transform $x = x_0 + \alpha^{mM}X$, $y = \alpha^{-mMt}v$, where $t = 1/(r-1) - n + 1 > 0$, f is analytic and nonzero in x_0, M is a natural number such that tM is integer, and obtain a reduced equation for $\alpha = 0$ equation

$$v^{(n)} = f(x_0)\left(v^{(n-1)}\right)^r.$$

This equation should admit no movable critical singularities. It holds only if $r = 1 + 1/\mu$, where μ is an integer number, not less than $n-1$, or if $r = 1$.

Consequently, $\sum_{j=1}^{s} p_j\beta_j = 1 + 1/\mu$ or $\sum_{j=1}^{s} p_j\beta_j = 1$, while for all j such that $\beta_j < 1$ we have $\beta_j = 1 - 1/\mu_j$, where $\mu_j \in N$. One can see that there are only the following possibilities (up to the order of P_j):

(1) $s = 1,\ p_1 = 1,\ \beta_1 = 1 + 1/\mu,\ \mu \geq n,\ \mu \in N$;
(2) $s = 1,\ p_1 = 1,\ \beta_1 = 1 - 1/\mu,\ \mu \geq n,\ \mu \in N$;
(3) $s = 1,\ p_1 = 2,\ \beta_1 = 1/2$;
(4) $s = 2,\ p_1 = p_2 = 1,\ \beta_1 = \beta_2 = 1/2$;
(5) $s = 2,\ p_1 = p_2 = 1,\ \beta_1 = 1/2,\ \beta_2 = 2/3,\ n \leq \mu = 6$;
(6) $s = 2,\ p_1 = p_2 = 1,\ \beta_1 = 1/2,\ \beta_2 = 3/4,\ n \leq \mu = 4$.

Note that case (4) is a subcase of 3. The corresponding Equations (1) sorted by the order are as follows:

$$y^{(n)} = f(x)\left(\sum_{j=0}^{n-1}(-1)^j h^{(j)}(x)y^{(n-j-1)} + \psi\right)^{1+1/\mu}, \quad \mu \geq n \tag{44}$$

$$y^{(n)} = f(x)\left(\sum_{j=0}^{n-1}(-1)^j h^{(j)}(x)y^{(n-j-1)} + \psi\right)^{1-1/\mu}, \quad \mu > 1 \tag{45}$$

$$y^{(n)} = f(x)S\left(l_{n-1}(y^{(n-1)}, y^{(n-2)}, \ldots, y, x), \ldots, l_0(y^{(n-1)}, y^{(n-2)}, \ldots, y, x)\right)^{1/2}, \tag{46}$$

$$\begin{aligned} y^{(n)} = {} & f(x)\left(\sum_{j=0}^{n-1}(-1)^j h_1^{(j)}(x)y^{(n-j-1)} + \psi_1\right)^{1/2} \\ & \times\left(\sum_{j=0}^{n-1}(-1)^j h_2^{(j)}(x)y^{(n-j-1)} + \psi_2\right)^{2/3}, \quad n = 4, 5, 6, \end{aligned} \tag{47}$$

$$\begin{aligned} y^{(IV)} = {} & f(x)\left(h_1(x)y''' - h_1'(x)y'' + h_1''(x)y' - h_1'''(x)y + \psi_1\right)^{1/2} \\ & \times\left(h_2(x)y''' - h_2'(x)y'' + h_2''(x)y' - h_2'''(x)y + \psi_2\right)^{3/4}, \end{aligned} \tag{48}$$

where h, h_1, h_2 are arbitrary polynomials with constant coefficients of order not higher than $n - 1$, ψ, ψ_1, ψ_2 are complex constants, S denotes a square form of all variables with constant coefficients, μ is a natural number.

Remark 1: While writing down the equations (44)–(48) we used the following representation

$$\begin{aligned} & L\left(l_{n-1}\left(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\right), \ldots, l_0\left(y^{(n-1)}, y^{(n-2)}, \ldots, y, x\right)\right) \\ & \quad = \sum_{j=0}^{n-1}(-1)^j h^{(j)}(x)y^{(n-j-1)} + \psi \end{aligned}$$

that can be easily obtained for arbitrary linear forms L with constant coefficients; here $h(x)$ is a certain polynomial of degree not higher than $n-1$ and ψ is a complex constant.

THEOREM 1. *If the Equation (1) of order $n \geq 4$ is free of movable critical singularities, then it should be of one of the following six forms: (42), (44)–(48).*

For instance, all of Equations (1) of order $n = 4$ free of movable critical singularities belong to one of the following six classes:

$$y^{(IV)} = f(x)\big((ax^3+bx^2+cx+d)y''' - (3ax^2+2bx+c)y'' + (6ax+2b)y' - 6ay + \psi\big)^{1+1/\mu}, \ \ \mu \geq 4 \tag{49}$$

$$y^{(IV)} = f(x)\big((ax^3+bx^2+cx+d)y''' - (3ax^2+2bx+c)y'' + (6ax+2b)y' - 6ay + \psi\big)^{1-1/\mu}, \ \ \mu > 1 \tag{50}$$

$$y^{(IV)} = f(x)S\big(x^3y''' - 3x^2y'' + 6xy' - 6y, x^2y''' - 2xy'' + 2y', xy''' - y'', y'''\big)^{1/2}, \tag{51}$$

$$\begin{aligned} y^{(IV)} = {} & f(x)\big((a_1x^3+b_1x^2+c_1x+d_1)y''' - (3a_1x^2+2b_1x+c_1)y'' \\ & + (6a_1x+2b_1)y' - 6a_1y + \psi_1\big)^{1/2} \\ & \cdot \big((a_2x^3+b_2x^2+c_2x+d_2)y''' - (3a_2x^2+2b_2x+c_2)y'' \\ & + (6a_2x+2b_2)y' - 6a_2y + \psi_2\big)^{2/3}, \end{aligned} \tag{52}$$

$$\begin{aligned} y^{(IV)} = {} & f(x)\big((a_1x^3+b_1x^2+c_1x+d_1)y''' - (3a_1x^2+2b_1x+c_1)y'' \\ & + (6a_1x+2b_1)y' - 6a_1y + \psi_1\big)^{1/2} \\ & \cdot \big((a_2x^3+b_2x^2+c_2x+d_2)y''' - (3a_2x^2+2b_2x+c_2)y'' \\ & + (6a_2x+2b_2)y' - 6a_2y + \psi_2\big)^{3/4}, \end{aligned} \tag{53}$$

$$\begin{aligned} y^{(IV)} = {} & \big((ax^3+bx^2+cx+d)y''' - (3ax^2+2bx+c)y'' \\ & + (6ax+2b)y' - 6ay + \psi\big)^{1/2} \\ & \cdot \big(A(x)y'' + B(x)y'y + C(x)y^3 + D(x)y' + E(x)y^2 + F(x)y + G(x)\big), \end{aligned} \tag{54}$$

where $a, b, c, d, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, \psi_1, \psi_2$ are complex numbers, S denotes a square form of all variables with constant coefficients, μ is a natural number, A, B, C, D, E, F, G are locally analytic functions in x.

Remark 2: For high-order Equations (1) with $n \geq 5$ only four cases (44)–(47) are possible, while for $n \geq 7$ only three cases (44)–(46) are possible.

Furthermore we consider each of these six Equations (42), (44)–(48) and obtain necessary and sufficient conditions of freedom from movable critical singularities for them.

6. Equations (42), (44)–(48)

6.1. Equations (44) and (45)

In both equations let $u = \sum_{j=0}^{n-1}(-1)^j h^{(j)}(x)y^{(n-j-1)} + \psi$ and note that $u' = h(x)y^{(n)}$. So the Equations (44) and (45) will be transformed to

$$u' = f(x)h(x)u^{1+1/\mu} \text{ and } u' = f(x)h(x)u^{1-1/\mu}$$

correspondingly. These equations admit general solutions $u = (-\mu\int(f(x)h(x))\,dx + C)^{-\mu}$ and $u = (\mu\int(f(x)h(x))\,dx + C)^{\mu}$ correspondingly, where C is an arbitrary complex constant. The initial variable y can be found as $y = \int\int\ldots\int(u'/h(x))\,dx^n$. In the second case, corresponding to Equation (45), u as well as y certainly does not admit any movable singularities.

For Equation (44) function u admits movable poles, while y, being expressed in the form $y = \int\int\ldots\int[\mu^2 f(x)(-\mu\int(f(x)h(x))\,dx + C)^{-\mu-1}]\,dx^n$ can admit either movable poles or movable logarithmic branch points. Assume $g(x) = -\mu\int(f(x)h(x))\,dx$ and let $C = -g(x_0)$, where x_0 is an arbitrary complex number, and $F(x, x_0) = f(x)(\frac{g(x)-g(x_0)}{x-x_0})^{-\mu-1}$. Then one can obtain the condition of freedom from movable critical singularities for Equation (44) in the following form: $\frac{\partial^k F}{\partial x^k}(x, x_0)|_{x=x_0} \equiv 0$ for $k = \mu - n + 1, \mu - n + 2, \ldots, \mu$. Note that $\frac{\partial^k F}{\partial x^k}(x, x_0)|_{x=x_0} = [f(x)(g'(x))^{-\mu-1}]^{(k)}|_{x=x_0} = [(-\mu)^{-\mu-1}f(x)^{-\mu}h(x)^{-\mu-1}]^{(k)}|_{x=x_0}$. So the above condition is equivalent to $f(x)^{-\mu}\,h(x)^{-\mu-1} = q_{\mu-n}(x)$, where $q_{\mu-n}$ is an arbitrary polynomial of degree not higher than $\mu - n$. The condition can be rewritten in the form $f(x) = (q_{\mu-n}(x))^{-1/\mu}\,h(x)^{1+1/\mu}$.

6.2. The Equation (46)

The Equation (46) does not admit movable singularities, because one can see that y satisfies a linear differential system

$$\begin{cases} y^{(n)} = f(x)u, \\ u' = \Omega(y^{(n-1)}, y^{(n-2)}, \ldots, y, x), \end{cases}$$

where $u = S(l_{n-1}(y^{(n-1)}, y^{(n-2)}, \ldots, y, x), \ldots, l_0(y^{(n-1)}, y^{(n-2)}, \ldots, y, x))^{1/2}$ and Ω is a linear form in y and its derivatives with coefficients, locally analytic in x.

6.3. The Equations (47) and (48)

The Equations (47) and (48) may be united into an equation

$$y^{(n)} = f(x)\left(\sum_{j=0}^{n-1}(-1)^j h_1^{(j)}(x)y^{(n-j-1)} + \psi_1\right)^{1/2} \times \left(\sum_{j=0}^{n-1}(-1)^j h_2^{(j)}(x)y^{(n-j-1)} + \psi_2\right)^{r}, \qquad (55)$$

where h_1, h_2 are arbitrary polynomials with constant coefficients of order not higher than $n - 1$, ψ_1, ψ_2 are complex constants, $r = 2/3$ or $3/4$. Let $u = (\sum_{j=0}^{n-1}(-1)^j h_1^{(j)}(x)y^{(n-j-1)} + \psi_1)^{1/2}$, $v = (\sum_{j=0}^{n-1}(-1)^j h_2^{(j)}(x)y^{(n-j-1)} + \psi_2)^{1/\mu}$. Then $u' = h_1(x)y^{(n)}/u$, $v' = h_2(x)y^{(n)}/v^{\mu-1}$. One can obtain a differential system

$$\begin{cases} u' = f(x)h_1(x)v^{\mu-1}, \\ v' = f(x)h_2(x)u. \end{cases}$$

This system is free of movable critical singularities if and only if $h_1(x)/h_2(x) \equiv const$. This condition is equivalent to $L_2 \equiv aL_1 + b$, where a and b are constants. In this case function y is linked with u or v by a second-order linear differential equation. The conditions of freedom from movable critical singularities for y can be obtained analogously like it has been done for Equation (44).

6.4. The Equation (42)

Equation (42) has been considered in [15].

For Equation (42) one can obtain the following reduced equation

$$v^{(IV)} = \sqrt{h(x_0)}\sqrt{v'''}(a(x_0)v'' + b(x_0)vv' + c(x_0)v^3),$$

where x_0 is an arbitrary complex constant. One can see that if this reduced equation is free of movable critical singularities, then $b(x_0) = c(x_0) = 0$. Because x_0 is arbitrary, we have $b(x) \equiv c(x) \equiv 0$. Then Equation (42) may be rewritten in the following form

$$\frac{d\sqrt{h(x)y''' - h'(x)y'' + h''(x)y' + h'''(x)y + \psi}}{dx} = \frac{\sqrt{h(x)}}{2}\left(a(x)y'' + d(x)y' + e(x)y^2 + f(x)y + g(x)\right). \qquad (56)$$

Assume $\tilde{a}(x) = a(z)\sqrt{h(x)}/2, \tilde{d}(x) = d(z)\sqrt{h(x)}/2, \tilde{f}(x) = f(z)\sqrt{h(x)}/2$, $\tilde{g}(x) = g(z)\sqrt{h(x)}/2$. Using the standard Painlevé test technique one can obtain the following necessary conditions for freedom from movable critical singularities:

$$e(x) \equiv 0,\ \tilde{g}(x) = \tilde{d}'(x) - \tilde{a}''(x),\ \tilde{d}(x) \equiv 2\tilde{a}'(x) - 2\tilde{a}(x)h'(x)/(3h(x))\,. \quad (57)$$

If these conditions hold, Equation (56) admits the first integral of the form:

$$\begin{aligned} &h(x)y''' - h'(x)y'' + h''(x)y' + h'''(x)y + \psi \\ &= \left(\tilde{a}(x)y' + \left(\tilde{a}'(x) - \frac{2\tilde{a}(x)h'(x)}{3h(x)}\right)y + \int \tilde{f}(x)\,dx + C\right)^2, \end{aligned} \quad (58)$$

where C is an arbitrary complex constant. Apply the transformation $y = K\frac{h(x)^{2/3}}{\tilde{a}(x)}u + \xi(x, C)$, $X = \frac{K}{6}\int \frac{\tilde{a}(x)}{h(x)^{1/3}}\,dx$, where K is a complex constant, ξ is a certain function locally analytic in x, C. For the certain function ξ one can obtain an equation of the form

$$u''' = \lambda(X)u'' + 6(u')^2 + \theta(X)u + \omega(X, C).$$

According to Chazy [5], this equation is free of movable critical singularities if and only if $\lambda(X) \equiv 0$, $\theta''(X) \equiv 0$, $\partial^2\omega(X, C)/\partial X \equiv \theta(X)^2/6$. Under these conditions the given equation is solvable in terms of elliptical functions or first Painlevé transcendent. The above conditions, together with (57) and $b(z) \equiv c(z) \equiv 0$ are equivalent to an alternative of the two following series of conditions: either

$$\begin{aligned} &h(x) = (x+T)^3,\ a(x) = A/(x+T)^2,\ d(x) = -A/(x+T)^3, \\ &f'''(x) = 0,\ b(x) = c(x) = e(x) = 0, \end{aligned}$$

where A are T are complex constants, or

$$h'(x) = a'(x) = f'''(x) = 0,\ b(x) = c(x) = d(x) = e(x) = 0.$$

for all x. So the initial Equation (42) is free of movable critical singularities if and only if the above alternative of conditions holds.

7. Conclusion

We proved that all binomial type ordinary differential Equations (1) of order $n \geq 4$ free of movable critical singularities are of the following forms: (44)–(48) and (42). Necessary and sufficient conditions of freedom from movable critical singularities for these six equations are obtained. Under these conditions Equations (44)–(48) appear to be either integrable in quadratures either linearizable, while Equation (42) can be solved in terms of elliptical functions or first Painlevé transcendent.

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